2021

PHYSICS — HONOURS

Paper : DSE-A-1(a)

(Advanced Mathematical Methods Theory)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer ***question nos.*** **1** and **2**, and ***any four*** questions from the rest (**Q. 3** to **Q. 8**).

1. Answer ***any five*** from the following : 2×5

(a) Show that the inverse of a linear operator is also a linear operator.

(b) Define a Unitary operator. Show that transformation by a unitary operator preserves the inner product of the vectors.

(c) Use Gram-Schmidt process to transform the basis vectors $u_1 = (1, 1, 1)$, $u_2 = (-1, 1, 0)$, $u_3 = (1, 2, 1)$ into an orthogonal basis $\{v_1, v_2, v_3\}$ assuming standard Euclidean inner product.

(d) The set of all real triplets (x, y, z) forms a vector space. Check whether the mapping $(x, y, z) \rightarrow (x, y, 0)$ is a linear transformation or not.

(e) When a pair of elements of a group is said to be conjugate to each other? Define class of a group.

(f) Show that a second rank contravariant symmetric tensor remains symmetric under a general coordinate transformation.

(g) Show that the SU(2) group has only three independent parameters.

2. Answer ***any three*** questions :

(a) Find g^{ij} and $g \equiv \det(g^{ij})$ corresponding to the metric tensor

$$ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6dx^1dx^2 + 4dx^2dx^3$$ 5

(b) Define projection operators. Prove that projection operators P are pairwise orthogonal *i.e.* $P_iP_j = 0$ if $i \neq j$ and $P_i^2 = P_i$. Show that it can only have eigenvalues 0 and 1. 1+2+2

(c) Using the properties of the Levi-Civita tensor ϵ_{ijk} show that

(i) $\vec{A}.(\vec{A}\times\vec{B}) = 0$

(ii) $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \nabla^2\vec{A}$ 2+3

(d) Show that the matrices : $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ form a basis for the vector space formed by the set of all 2×2 real, symmetric matrices. 5

(e) Consider two groups G and G'. The group G consists of four elements $\{1, i, -1, -i\}$ with ordinary multiplication as the rule of combination. The elements of the other group G' are the following four matrices with matrix multiplication as the rule of combination.

$$A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, C=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, D=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

Using group multiplication tables, show that G and G' are isomorphic. 5

3. (a) How do we define dimension of a linear vector space? Define inner product space. When do we call an inner product space to be complete?

(b) From Cauchy-Schwarz inequality $|\langle U|W\rangle| \le |U||W|$, prove the Triangle inequality $|U+W| \le |U|+|W|$ where $|U\rangle$ and $|W\rangle$ are two non-zero vectors in an inner product space, and for any vector $|A\rangle$, $|A| = \sqrt{\langle A\,|\,A\rangle}$.

(c) When do we call the eigenvalues of an operator to be degenerate? Show that two commuting Hermitian operators possess a set of common eigenvectors. Assume the eigenvalues are non-degenerate. (1+2+2)+2+(1+2)

4. (a) Let the matrix representation of an operator T on V be of the form : $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$ with respect to a set of basis $\hat{e}_1, \hat{e}_2, \hat{e}_3$. How does the representation changes in a new set of basis—

$$\hat{f}_1 = \frac{1}{\sqrt{2}}(\hat{e}_1 + \hat{e}_2),\ \hat{f}_2 = \frac{1}{\sqrt{2}}(-\hat{e}_1 + \hat{e}_2),\ \hat{f}_3 = \hat{e}_3\,?$$

(b) What do you mean by a Normal operator? Given that A is a Normal matrix, its eigenvalues λ_j are in general complex. Show that $\mathrm{Re}(\lambda_j)$ and $\mathrm{Im}(\lambda_j)$ are eigenvalues of $(A + A^\dagger)/2$ and $(A - A^\dagger)/2$ respectively.

(c) Show that the eigenvalues of a Hermitian operator are real and the eigenvectors belonging to different eigenvalues are mutually orthogonal. 4+(1+2)+3

5. (a) Two adjacent edges of a uniform square plate of mass M and side a are chosen as the x and y axes of a three dimensional Cartesian coordinate system. Find the inertia tensor for the plate with respect to the axes chosen. Find the principal moments of inertia.

(b) The moment of inertia tensor of a body is $\begin{pmatrix}I_1 & 0 & 0\\ 0 & I_2 & 0\\ 0 & 0 & I_3\end{pmatrix}$. Prove that if $I_1 = I_2$, then the moment of inertia of the body about any axis in the $x-y$ plane, passing through the origin is the same.

(5+3)+2

6. (a) Show that the familiar Kronecker delta δ_{kl} is really a mixed tensor of rank two δ^k_l . Why is it called an isotropic tensor?

(b) (i) The field strength tensor $F_{\mu\nu}$ is defined by $\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$ where A_μ is the four-vector potential. Express the components of $F_{\mu\nu}$ in terms of the electric and the magnetic field $\vec{E}$ and $\vec{B}$.

(ii) Given the components of $\vec{E}$ and $\vec{B}$ in a certain inertial frame S, find the components of $\vec{E}$ and $\vec{B}$ in another inertial frame S', moving with a uniform velocity v with respect to S along the common x-axis. (2+1)+(3+4)

7. (a) Identify the elements in the symmetry group of a rectangle. Hence construct the multiplication table for this group.

(b) Is this group Abelian?

(c) Identify any two subgroups of this group. (2+5)+1+2

8. (a) Show that the group generated by two commuting elements A and B such that $A^2 = B^3 = E$, is cyclic.

(b) Justify that SO(2), the group that describes rotational symmetry about a single axis, is an example of a Lie group. Show that the generator of this group is one of the Pauli matrices.

(c) Consider the Lie algebra with basis $\{e_1, e_2, e_3\}$ and the commutators

$[e_1, e_2] = e_3, \; [e_2, e_3] = e_1, [e_3, e_1] = e_2.$

Find the adjoint representation. 3+(2+2)+3